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# GROUP PURSUIT UNDER BOUNDED EVADER COORDINATES*

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Effective methods are proposed for solving the group pursuit problem with constraints on the evader's state. The paper is closely related to the investigations in /1-4/ (**) and is a development of the results in /5/ in the case of arbitrary linear equations of motion of the evader.

1. Given the differential game

$$z_i^{\cdot} = A_i z_i + \varphi_i(u_i, v), \quad z_i \in E^{n_i}, \quad u_i \in U_i, \; v \in V, \; i \in N_m = \{1, \ldots, m\} \tag{1.1}$$

where $E^{n_i}$ is a finite-dimensional Euclidean space, $A_i$ are square matrices of order $n_i$, $U_i$, $V$ are nonempty compacta, the functions $\varphi_i(u_i, v)$ are continuous along variables collection. The terminal set $M$ consists of sets $M_i^*$, $i \in N_m$, of form $M_i^* = M_i^\circ + M_i$, where $M_i^\circ$ are linear subspaces of $E^{n_i}$, while $M_i$ are closed convex sets from the orthogonal complements $L_i$ to $M_i^\circ$ in space $E^{n_i}$, and for $i \in N_m \setminus N_k$, $k \leqslant m$, $M_i = \{a_i\}$, $a_i$ is some vector from $L_i$. Game (1.1) is considered ended if for some $t > 0$ we have $z_i(t) \in M_i^*$ for at least one $i$.

We say that the differential game (1.1) can be ended from a prescribed position $z^\circ = (z_1^\circ, \ldots, z_m^\circ)$ no later than by the time $T = T(z^\circ)$ if measurable functions $u_i(t) = u_i(z_i^\circ, v(t)) \in U_i$, $t \in [0, T]$, exist such that the solutions of the equations

$$z_i^{\cdot} = A_i z_i + \varphi_i(u_i(t), \; v(t)), \quad z_i(0) = z_i^\circ, \quad i \in N_m$$

for some $i = i(v(\cdot))$, hit onto set $M_i$ no later than at the instant $t = T$ for any measurable functions $v(\cdot) = \{v(t)\colon v(t) \in V, \; t \in [0, T]\}$. Here the pursuers can use not only the instantaneous values of the evader's control but also the entire previous history $v(s)$, $s \in [0, t]$.

2. Let $\pi_i$ be the operator of orthogonal projection from $E^{n_i}$ onto a subspace $L_i$. Consider the many-valued mappings

$$\Phi_i(t, v) = \pi_i \exp(tA_i) \varphi_i(U_i, \varepsilon_i(t) v).$$

$$\Phi_i(t) = \bigcap_{v \in V} \Phi_i(t, v), \quad i \in N_k, \quad t \geqslant 0$$

where $\exp(tA_i)$ is the fundamental matrix of the system $z_i^{\cdot} = A_i z_i$, and $\varepsilon_i(t)$ are certain measurable functions taking values from the interval /0,1/. Let measurable functions $\varepsilon_i(t)$, $i \in N_k$, and a number $T_0 > 0$, $\varepsilon_i(t) \in [0, 1]$, $t \in [0, T_0]$, exist such that the following conditions are fulfilled.

Condition 1. The sets $\Phi_i(t)$ are nonempty for all $i \in N_k$, $T_0 \geqslant t \geqslant 0$. We set

$$\varphi_i^*(t, u_i, v) = \varphi_i(u_i, v) - \varphi_i(u_i, \varepsilon_i(t) v)$$

Condition 2. The sets

$$W_i(t) = M_i \overset{*}{-} \int_0^t \pi_i \exp((t-\tau) A_i) \varphi_i^*(t-\tau, U_i, V)\, d\tau$$

are nonempty for all $i \in N_k$, $T_0 \geqslant t > 0$ ($\overset{*}{-}$ is the operation of geometric subtraction of sets /6/).

Having fixed certain measurable selectors $\varphi_i(t) \in \Phi_i(t)$, we set

$$\xi_i(t, z_i) = \pi_i \exp(tA_i) z_i + \int_0^t \varphi_i(t-\tau)\, d\tau, \quad i \in N_k, \quad t \in [0, T_0]$$

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*) Prikl.Matem.Mekhan.,46,No.6,pp.906-913,1982

**) Also see: Chentsov A.G., On certain aspects of the structure of differential encounter-evasion games. Sverdlovsk, 1979, Deposited in VINITI, No.205-80,1980.

and denote

$$\alpha_i(t,\tau,z_i,v)=\begin{cases}\max(\alpha\geqslant 0:\ \{\Phi_i(t-\tau,v)-\varphi_i(t-\tau)\}\cap\\ \{\alpha(W_i(t)-\xi_i(t,z_i))\}\neq\varnothing),\ \xi_i(t,z_i)\overline{\in} W_i(t)\\ t^{-1},\quad \xi_i(t,z_i)\in W_i(t)\end{cases} \tag{2.1}$$
$$i\in N_k,\ T_0\geqslant t\geqslant\tau>0,\ v\in V$$

Lemma 1. Let Conditions 1 and 2 be fulfilled, the mappings $\varphi_i(U_i, \varepsilon_i(t)v)$ be convex-valued, $\varphi_i(t)$ be a measurable selector of mapping $\Phi_i(t)$, $v\in V$. Then, if $\xi_i(t, z_i)\overline{\in} W_i(t)$, then

$$\alpha_i(t,\tau,z_i,v)=\inf_{p\in P_i(t,z_i)}\{C_{\Phi_i(t-\tau,v)}(-p)+(p,\varphi_i(t-\tau))\},$$
$$T_0\geqslant t\geqslant\tau\geqslant 0,\quad i\in N_k \tag{2.2}$$

where $P_i(t,z_i)=\{p\in L_i: -C_{W_i(t)}(p)+(p,\xi_i(t,z_i))=1\}$, and $C_{\Phi_i(t,v)}(p)$, $C_{W_i(t)}(p)$ are the support functions of the corresponding sets.

Proof. From Condition 1 follows the inclusion

$$0\in\Phi_i(t-\tau,v)-\varphi_i(t-\tau)$$

for all $v\in V$, $T_0\geqslant t\geqslant\tau\geqslant 0$, which is equivalent to the inequality

$$C_{\Phi_i(t-\tau,v)}(-p)+(p,\varphi_i(t-\tau))\geqslant 0\ \ \forall p\in L_i \tag{2.3}$$

From the property of the geometric subtraction operation it follows that the mapping $W_i(t)$ is convex-valued /6/. The emptiness of the intersection in expression (2.1) is equivalent to the inequality /7/

$$C_{\Phi_i(t-\tau,v)}(-p)+(p,\varphi_i(t-\tau))\geqslant\alpha((p,\xi_i(t,z_i))-C_{W_i(t)}(p))\ \forall p\in L_i$$

When $(p,\xi_i(t,z_i))-C_{W_i(t)}(p)\leqslant 0$ the last inequality is fulfilled for any nonnegative $\alpha$ since (2.3) holds. If, however, $(p,\xi_i(t,z_i))-C_{W_i(t)}(p)>0$, then, having set $(p,\xi_i(t,z))-C_{W_i(t)}(p)=1$, we obtain

$$C_{\Phi_i(t-\tau,v)}(-p)+(p,\varphi_i(t-\tau))\geqslant\alpha$$

Hence follows formula (2.2). The following condition is assumed fulfilled for $i\in N_m\setminus N_k$

Condition 3. The sets $\pi_i\exp(tA_i)\varphi_i(U_i,v)$, $i\in N_m\setminus N_k$, consist of unique points $\varphi_i(t,v)$ for fixed $t, v$, $T_0\geqslant t\geqslant 0$, $v\in V$. For $i\in N_m\setminus N_k$ we set

$$\xi_i(t,z_i)=\pi_i\exp(tA_i)z_i \tag{2.4}$$
$$\alpha_i(t,\tau,z_i,v)=\begin{cases}\alpha:\ \alpha(a_i-\xi_i(t,z_i))=\varphi_i(t-\tau,v),\quad a_i\neq\xi_i(t,z_i)\\ \|\varphi_i(t-\tau,v)\|+t^{-1},\quad a_i=\xi_i(t,z_i)\end{cases}$$
$$T_0\geqslant t\geqslant\tau>0,\quad v\in V$$

We denote

$$\lambda(t,z)=1-\inf_{v(\cdot)}\max_{i\in N_m}\int_0^t\alpha_i(t,\tau,z_i,v(\tau))\,d\tau$$
$$T(z)=\{t>0:\ \lambda(t,z)=0\}$$

where $v(\cdot)$ is a function measurable on the interval $[0, t]$ taking values from set $V$.

Theorem 1. Let Conditions 1–3 be fulfilled for differential game (1.1) and let $T(z^\circ)\leqslant T_0$. Then from a prescribed initial position $z^\circ$ it can be ended no later than by time $T(z^\circ)$.

Proof. Let $v(\tau)$, $v(\tau)\in V$, $\tau\in[0,T]$, $T=T(z^\circ)$ be some measurable function. We set

$$h(T,t,z^\circ,v(\cdot))=1-\max\Big\{\max_{i\in N_k}\int_0^t\alpha_i(T,\tau,z_i^\circ,v(\tau))\,d\tau,\ \max_{i\in N_m\setminus N_k}\int_0^t\alpha_i(t,\tau,z_i^\circ,v(\tau))\,d\tau\Big\}$$

Since $h(T,0,z^\circ,v(\cdot))=1$, while for $i\in N_m\setminus N_k$, $a_i\neq\xi_i(t,z_i^\circ)$ the function $\alpha_i(t,\tau,z_i^\circ,v)$ depends continuously on $t$, we have that $h(T,t,z^\circ,v(\cdot))$ depends continuously on $t$ and from the definition of function $\lambda(t,z)$ it follows that an instant $t_*$, $0<t_*\leqslant T$, exists such that $h(T,t_*,v(\cdot))=0$.

Let us indicate a method for choosing the controls for $i\in N_k$. Let $\xi_i(T,z_i^\circ)\overline{\in} W_i(T)$. Then for $0\leqslant\tau<t_*$ we choose the control $u_i(\tau)\in U_i$ and the function $\varkappa_i(\tau)\in W_i(T)$ from the equation

$$\pi_i \exp((T-\tau)A_i)\,\varphi_i(U_i(\tau),\ \varepsilon_i(T-\tau)v(\tau)) - \varphi_i(T-\tau) = -\alpha_i(T,\ \tau,\ z_i^\circ,\ v(\tau))(\varkappa_i(\tau) - \xi_i(T,\ z_i^\circ)) \tag{2.5}$$

The function $\alpha_i(T, \tau, z_i^\circ, v(\tau))$ is measurable in $\tau$; therefore, on the strength of the Filippov–Castaing theorem /8/, the solvability of Eq.(2.5) in the class of measurable functions $u_i(\tau)$, $\varkappa_i(\tau)$, $0 \leqslant \tau < t_*$ follows from Conditions 1 and 2. For $t_* \leqslant \tau \leqslant T$ we set $\alpha_i(T, \tau, z_i^\circ, v) \equiv 0$ and we choose the control $u_i(\tau)$ from the resulting Eq.(2.5). If $\xi_i(T, z_i^\circ) \in W_i(T)$ then we set $\varkappa_i(\tau) \equiv \xi_i(T, z_i^\circ)$ and we choose the control $u_i(\tau)$ from Eq.(2.5) with a zero right-hand side. The representation

$$\pi_i z_i(t) = \pi_i \exp(tA_i) z_i^\circ + \int_0^t \pi_i \exp((t-\tau)A_i)\,\varphi_i(u_i(\tau), v(\tau))\,d\tau, \quad i \in N_m \tag{2.6}$$

follows from the Cauchy formula. If $h(T, t_*, z^\circ, v(\cdot)) = 0$, then a number $j$ exists such that one of the following equalities is fulfilled:

$$1 - \int_0^{t_*} \alpha_j(T, \tau, z_j^\circ, v(\tau))\,d\tau = 0, \quad j \in N_k \tag{2.7}$$

$$1 - \int_0^{t_*} \alpha_j(t_*, \tau, z_j^\circ, v(\tau))\,d\tau = 0, \quad j \in N_m \setminus N_k \tag{2.8}$$

Let $j \in N_k$. Then, by adding and subtracting the quantities

$$\int_0^T \pi_j \exp((T-\tau)A_j)\,\varphi_j(u_j(\tau),\ \varepsilon_j(T-\tau)v(\tau))\,d\tau,\ \int_0^T \varphi_j(T-\tau)\,d\tau$$

from both sides of equality (2.6) with $i = j$, $t = T$, as well as taking into account the control selection law, we obtain

$$\pi_j z_j(T) = \xi_j(T, z_j^\circ)\Big(1 - \int_0^T \alpha_j(T, \tau, z_j^\circ, v(\tau))\,d\tau\Big) + \int_0^T \alpha_j(T, \tau, z_j^\circ, v(\tau))\,\varkappa_j(\tau)\,d\tau + \int_0^T \pi_j \exp((T-\tau)A_j)\,\varphi_j^*(T-\tau, u_j(\tau), v(\tau))\,d\tau$$

Hence with due regard to formulas (2.5), (2.7), to the convex-valuedness of mapping $W_j(T)$ and to the property of the geometric subtraction operation, we obtain $\pi_j z_j(T) \in M_j$. Let $j \in N_m \setminus N_k$. Let us consider the case when $a_j \neq \xi_j(t_*, z_j^\circ)$. By virtue of equality (2.8) and Condition 3, from (2.4) we have

$$a_j - \xi_j(t_*, z_j^\circ) - \int_0^{t_*} \varphi_j(t_* - \tau, v(\tau))\,d\tau = 0$$

or $a_j = \pi_j z_j(t_*)$. If $a_j = \xi_j(t_*, z_j^\circ)$, then from equality (2.8) we obtain

$$\int_0^{t_*} \|\varphi_j(t-\tau, v(\tau))\|\,d\tau = 0 \ \text{ or } \ \int_0^{t_*} \varphi_j(t_* - \tau, v(\tau))\,d\tau = 0$$

Hence with due regard to the initial assumption and to formula (2.6) we obtain $a_j = \pi_j z_j(t_*)$.

3. We fix certain measurable selectors $\varkappa_i(t)$ of the many-valued mappings $W_i(t)$, $i \in N_k$, $t \in [0, T_0]$ and we set

$$\eta_i(t, z_i) = \xi_i(t, z_i) - \varkappa_i(t)$$

We denote

$$\beta_i(t, \tau, z_i, v) = \begin{cases} \max(\beta \geqslant 0:\ -\beta\eta_i(t, z_i) \in \Phi_i(t-\tau, v) - \varphi_i(t-\tau)), & \eta_i(t, z_i) \neq 0 \\ t^{-1}, & \eta_i(t, z_i) = 0 \end{cases}$$

$$i \in N_k, \quad T_0 \geqslant t \geqslant \tau > 0, \quad v \in V$$

**Lemma 2.** Let Conditions 1 and 2 be fulfilled, the mappings $\varphi_i(U_i, \varepsilon_i(t)v)$ be convex-valued, $\varphi_i(t)$ and $\varkappa_i(t)$ be measurable selectors of mappings $\Phi_i(t)$ and $W_i(t)$, respectively. Then, if $\eta_i(t, z_i) \neq 0$, then

$$\beta_i(t, \tau, z_i, v) = \inf_{\substack{p \in L_i \\ (p, \eta_i(t, z_i)) = 1}} \{C_{\Phi_i(t-\tau, v)}(-p) + (p, \varphi_i(t-\tau))\},$$
$$i \in N_k,\ T_0 \geqslant t \geqslant \tau > 0,\ v \in V$$

The proof is analogous to that of Lemma 1.

For $i \in N_m \setminus N_k$ we set $\xi_i(t, z_i) \equiv \eta_i(t, z_i)$, $\beta_i(t, \tau, z_i, v) \equiv \alpha_i(t, \tau, z_i, v)$. We denote

$$\mu(t, z) = 1 - \inf_{v(\cdot)} \max_{i \in N_m} \int_0^t \beta_i(t, \tau, z_i, v(\tau))\, d\tau$$
$$\Theta(z) = \{t > 0:\ \mu(t, z) = 0\}$$

**Theorem 2.** Let Conditions 1–3 be fulfilled for the differential game (1.1) and let $\Theta(z^\circ) \leqslant T_0$. Then from a prescribed initial position $z^\circ$ it can be ended no later than by time $\Theta(z^\circ)$.

The proof is carried out by the scheme used to prove Theorem 1.

4. Let $\omega_i(t, \tau)$, $i \in N_k$, $t \geqslant \tau \geqslant 0$, be certain numerical functions. We consider the many-valued mappings

$$F_i(t, \tau, U_i, v) = \Phi_i(t-\tau, v) - \omega_i(t, \tau) W_i(t)$$
$$F_i(t, \tau) = \bigcap_{v \in V} F_i(t, \tau, U_i, v),\quad t \geqslant \tau \geqslant 0,\quad i \in N_k$$

Let measurable functions $\varepsilon_i(t) \in [0, 1]$, measurable nonnegative functions $\omega_i(t, \tau)$ and a number $T_0$, $i \in N_k$, $T_0 \geqslant t \geqslant \tau \geqslant 0$, exist so as to fulfil the following condition:

**Condition 4.** The sets $F_i(t, \tau)$ are nonempty for all $i \in N_k$, $T_0 \geqslant t \geqslant \tau \geqslant 0$.

We fix certain measurable selectors $f_i(t, \tau)$ of mappings $F_i(t, \tau)$ and we set

$$\zeta_i(t, z_i) = \pi_i \exp(tA_i) z_i + \int_0^t f_i(t, \tau)\, d\tau$$

We denote

$$\gamma_i(t, \tau, z_i, v) = \begin{cases} \max(\gamma \geqslant 0:\ -\gamma\zeta_i(t, z_i) \in F_i(t, \tau, U_i, v) - f_i(t, \tau)) & \zeta_i(t, z_i) \neq 0 \\ t^{-1},\quad \zeta_i(t, z_i) = 0 \end{cases}$$
$$i \in N_k,\quad T_0 \geqslant t \geqslant \tau > 0,\quad v \in V$$

**Lemma 3.** Let Conditions 2 and 4 be fulfilled, mappings $\varphi_i(U_i, \varepsilon_i(t)v)$ be convex-valued, $f_i(t, \tau)$ be a measurable selector of mapping $F_i(t, \tau)$. Then, if $\xi_i(t, z_i) \neq 0$, then

$$\gamma_i(t, \tau, z_i, v) = \inf_{\substack{p \in L_i \\ (p, \zeta_i(t, z_i)) = 1}} \{C_{\Phi_i(t-\tau, v)}(-p) + (p, f_i(t, \tau)) +$$
$$\omega_i(t, \tau) C_{W_i(t)}(-p)\}\quad i \in N_k,\ T_0 \geqslant t \geqslant \tau > 0,\ v \in V$$

The proof is analogous to that of Lemma 1.

For $i \in N_m \setminus N_k$ we set $\zeta_i(t, z_i) \equiv \xi_i(t, z_i)$, $\gamma_i(t, \tau, z_i, v) \equiv \alpha_i(t, \tau, z_i, v)$. We denote

$$\nu(t, z) = 1 - \inf_{v(\cdot)} \max_{i \in N_m} \int_0^t \gamma_i(t, \tau, z_i, v(\tau))\, d\tau$$
$$\Gamma(z) = \{t > 0:\ \nu(t, z) = 0\}$$

**Theorem 3.** Let Conditions 2–4 be fulfilled for the differential game (2.1) and let $T = \Gamma(z^\circ) \leqslant T_0$ and

$$\int_0^T \omega_i(T, \tau)\, d\tau = 1,\quad i \in N_k$$

Then from a prescribed initial position $z^\circ$ it can be ended no later than by time $\Gamma(z^\circ)$.
The proof is based on the ideas used to prove Theorem 1.

5. Let $k = m = 1$. In all notation we omit the indices and we set $\varepsilon(t) \equiv 1$. Let us establish the connection between the pursuit plans presented in Sects. 2—4 and Pontriagin's first direct method /6/.

Corollary 1. Let Condition 1 be fulfilled. Then in order that

$$\pi \exp(tA) z \in M - \int_0^t \Phi(t-\tau)\, d\tau$$

it is necessary and sufficient that a measurable selector $\varphi(\tau) \in \Phi(\tau)$, $\tau \in [0, t]$, exist such that $\xi(t, z) \in M$.

From Corollary 1 it follows, in particular, that

$$T(z) \leqslant \Pi(z)$$

$$\Pi(z) = \left\{t > 0\colon \pi \exp(tA) z \in M - \int_0^t \Phi(t-\tau)\, d\tau\right\}$$

Corollary 2. Let Condition 1 be fulfilled. Then in order that

$$\left\{\pi \exp(tA) z + \int_0^t \Phi(t-\tau)\, d\tau\right\} \cap M \neq \varnothing$$

it is necessary and sufficient that a measurable selector $\varphi(\tau) \in \Phi(\tau)$, $\tau \in [0, t]$ and a vector $m \in M$ exist such that $\eta(t, z) = 0$.

Corollary 3. Let Condition 4 be fulfilled. Then in order that

$$-\pi \exp(tA) z \in \int_0^t F(t, \tau)\, d\tau \tag{5.1}$$

it is necessary and sufficient that a measurable selector $f(t, \tau) \in F(t, \tau)$, $\tau \in [0, t]$, exist such that $\zeta(t, z) = 0$. If furthermore

$$\int_0^t \omega(t, \tau)\, d\tau = 1$$

then the pursuit can be ended in time $t = t(z)$ prescribed by inclusion (5.1) from the initial position $z$.

The proofs of Corollaries 1—3 follow from the constructions in Sects.2—4. Thus, the plans in Sects.2 and 3 can, in particular, coincide with Pontriagin's first direct method, while the plan in Sect.4 leads to a certain modification of it.

6. Let us consider the problem of pursuing an evader by a group of controlled objects, in the situation when the evader cannot leave the confines of some open convex set, and let us show that it is a special case of differential game (1.1). The motions of the pursuers and the evader have the form

$$\begin{aligned} x_i^{\cdot} &= C_i x_i + u_i, \quad u_i \in U_i, \quad x_i \in E^{r_i}, \quad i \in N_k \\ y^{\cdot} &= By + v, \quad v \in V, \quad y \in E^s \end{aligned} \tag{6.1}$$

where certain coordinates of the evader are constrained:

$$G = \{y\colon (p_i, y) < l_i, \ \|p_i\| = 1, \ i \in N_m \setminus N_k\} \tag{6.2}$$

The sets $M_i^*$, $i \in N_k$, are prescribed just as in game (1.1), in the spaces $E^{n_i} = E^{r_i} \times E^s$. The pursuit process is considered ended if at least one of the pursuers catches the evader ($\{x_i, y\} \in M_i^*$ for some $i \in N_k$) or if the evader is forced to violate the constraints ($(p_i, y) = l_i$ for some $i \in N_m \setminus N_k$). We set

$$z_i = \{x_i, y\}, \quad A_i = \begin{Vmatrix} C_i & 0 \\ 0 & B \end{Vmatrix}, \quad \varphi_i(u_i, v) = \begin{Vmatrix} u_i \\ 0 \end{Vmatrix} + \begin{Vmatrix} 0 \\ v \end{Vmatrix}, \quad i \in N_k$$

$$z_i = y, \quad A_i = B, \quad \varphi_i(u_i, v) = v$$

$$M_i^\circ = \{z_i\colon (p_i, z_i) = 0\}, \quad a_i = l_i p_i, \quad i \in N_m \setminus N_k$$

By the same token the group pursuit problem (6.1) with constraints (6.2) is reduced to a constraint free problem of form (1.1). Such a reduction was used in /5/; an analogous method was applied earlier in /9/ for an escape problem. The very rigid Condition 2 was present in /5/, reducing the analysis essentially to a simple motion of the evader. Condition 3, replacing it, is fulfilled automatically.

7. Example 1. The pursuers and the evader move in accord with the equations

$$z_i^{\cdot} = az_i + u_i, \| u_i \| \leqslant 1, i \in N_k, z_i \in E^s$$
$$y^{\cdot} = ay + v, \| v \| \leqslant 1, y \in E^s$$

The set $M_i^*$ consists of points $\{z_i, y\}$, such that $\| z_i - y \| \leqslant \varepsilon_i$. The constraints on the evader's coordinates are

$$G = \{y \in E^s: (p_i, y) < l_i, p_i \in E^s, \| p_i \| = 1, i \in N_m \setminus N_k\}$$

We consider various cases.

$1^{\circ}$. $a < 0, \varepsilon_i > 0, i \in N_k$. We apply the plan in Sect.2. Condition 1 is fulfilled with $\varepsilon_i(t) \equiv 1$. Having set $\varphi_i(t) \equiv 0$, we obtain

$$\xi_i(t, z_i) = \exp(at) z_i, i \in N_k$$

Since $a < 0$, at the instant

$$t_i^* = a^{-1} \cdot \ln(\varepsilon_i \cdot \| z_i^{\circ} \|^{-1})$$

we have $\xi_i(t, z_i) \in M_i^*$. This goal is reached with the aid of the control $u_i(\tau) = v(\tau), \tau \in [0, t_i^*]$. Here the time $t_i^*$ coincides with the Pontriagin time by virtue of Corollary 1. Thus, each of the pursuers independently catch the evader in finite time from any initial positions, even without constraints (6.2).

$2^{\circ}$. $a < 0, \varepsilon_i = 0, i \in N_k$. From the method of invariant subspaces /10/ it follows that the evader can avoid capture in the case of one pursuer and without constraints. By virtue of Lemma 1 we have

$$\alpha_i(t, \tau, z_i^{\circ}, v) = \exp(-\tau a) \alpha_i(z_i^{\circ}, v)$$
$$\alpha_i(z_i^{\circ}, v) = \| z_i^{\circ} \|^{-2} ((v, z_i^{\circ}) + [(v, z_i^{\circ})^2 + \| z_i^{\circ} \|^2 (1 - \| v \|^2)]^{1/2}), \quad i \in N_k$$
$$\alpha_i(t, \tau, z_i^{\circ}, v) = \frac{(p_i, \exp(a(t-\tau)) v)}{l_i - (p_i, \exp(at) z_i^{\circ})}, \quad i \in N_m \setminus N_k$$

Let the phase constraints be a polyhedral cone ($l_i = 0$, $i \in N_m \setminus N_k$). We denote

$$\alpha(z^{\circ}) = \max_{i \in N_m} \min_{\| v \| \leqslant 1} \left\{ \alpha_i(z_i^{\circ}, v), \frac{(p_i, v)}{-(p_i, z_i^{\circ})} \right\}$$

Then the condition $\alpha(z^{\circ}) > 0$ is sufficient for completing the group pursuit, and the pursuit time is bounded by the quantity

$$-a^{-1} \ln \frac{\alpha(z^{\circ}) - a}{\alpha(z^{\circ})} \tag{7.1}$$

and the pursuers' controls are

$$u_i(\tau) = v(\tau) - \alpha_i(z_i^{\circ}, v(\tau)) z_i^{\circ}, i \in N_k, \tau \in [0, T(z^{\circ})]$$

$3^{\circ}$. $a > 0, \varepsilon_i = 0, i \in N_k, l_i = 0, i \in N_m \setminus N_k$. A sufficient condition for completion of pursuit is the condition $\alpha(z^{\circ}) > a$, and the pursuit time is bounded by quantity (7.1).

The case of simple motion ($a = 0$) was analyzed in /5/.

Example 2. (Pontriagin's check example with equal coefficients of friction). The motions of the pursuers and the evader are described by the equations

$$x_{1i}^{\cdot} = x_{2i}, x_{2i}^{\cdot} = ax_{2i} + u_i, x_{1i}, x_{2i} \in E^s, s \geqslant 2, \| u_i \| \leqslant 1, i \in N_k$$
$$y_1^{\cdot} = y_2, y_2^{\cdot} = ay_2 + v, y_1, y_2 \in E^s, \| v \| \leqslant 1, a < 0$$

The set $M_i^*$ consists of pairs $\{x_{1i}, y_1\}$, such that $x_{1i} = y_1$. The constraints on the evader's geometric coordinates $(y_1)$ are of form (6.2). We set

$$z_{1i} = x_{1i} - y, z_{2i} = x_{2i} - y, i \in N_k, z_{1i} = y_1, z_{2i} = y_2, i \in N_m \setminus N_k$$

We apply the plan in Sect.1. Condition 1 is fulfilled with $\varepsilon_i(t) \equiv 1, i \in N_k$. Here $\varphi_i(t) \equiv 0$, and

$$\xi_i(t, z_i^{\circ}) = z_{1i}^{\circ} + e_1(t) z_{2i}^{\circ}, i \in N_m$$

$e_1(t) = a^{-1}(1 - \exp(-at))$ /11/. On the strength of Lemma 1

$$\alpha_i(t, \tau, z_i^{\circ}, v) = e_1(t-\tau) \alpha_i(\xi_i(t, z_i^{\circ}), v), \xi_i(t, z_i^{\circ}) \neq 0, i \in N_k$$
$$\alpha_i(t, \tau, z_i^{\circ}, v) = \frac{(p_i, e_1(t-\tau) v)}{l_i - (p_i, y_1^{\circ} + e_1(t) y_2^{\circ})}, \quad \xi_i(t, z_i^{\circ}) \neq l_i p_i, \quad i \in N_m \setminus N_k$$

We denote

$$z_i{*} = z_{1i}{}^\circ + 1/az_{2i} = \lim_{t\to\infty} \xi_i(t, z_i{}^\circ), \quad i \in N_k$$

$$y{*} = y_1{}^\circ + 1/ay_2{}^\circ = \lim_{t\to\infty} \xi_i(t, z_i{}^\circ), \quad i \in N_m \diagdown N_k$$

$$\alpha(z^\circ) = \max_{i\in N_m} \min_{\|v\|\leqslant 1} \left\{\alpha_i(z_i{*}, v), \frac{(p_i, v)}{l_i - (p_i, y{*})}\right\}, \quad l_i \neq (p_i, y{*})$$

Condition $\alpha(z^\circ) > 0$ is sufficient for group pursuit completion. The pursuers' controls have the form indicated in /3/ (Example 3).

Examples 1 and 2 are solutions of problems of the type "cornered rat", "deadline game", "lion and man" /12/ in the formulation given.

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